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Structure Analyses for Large Scale Nonlinear Multipoint Boundary Value Problems

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1. INTRODUCTION

The main concern of this paper is with the structure analysis of large systems in nonlinear multipoint boundary value problems (MPBVPs). More specifically, it discusses systems of first-order nonlinear ordinary differential equations (ODEs) of the form

$$\begin{gathered} \dot{x}_i = f_i(x_1, x_2, ..., x_n, t), \qquad i = 1, 2, ..., n, \\ x = (x_1, x_2, ..., x_n)', \end{gathered} \tag{1.1}$$

with the general nonlinear boundary conditions (BCs) given by

$$\begin{gathered} g_i(x_1(t_1), ..., x_n(t_1), ..., x_j(t_l), ..., x_1(t_m), ..., x_n(t_m)) = 0, \\ m \geqslant 2, \quad i, j = 1, 2, ..., n, \quad l = 1, 2, ..., m, \end{gathered} \tag{1.2}$$

where f_i and g_i are twice continuously differentiable with respect to their arguments, f_i is continuous in t on $[t_1, t_m]$ and $'$ denotes transposition.

The MPBVP given by (1.1) and (1.2) can almost never be solved in a closed form. Usually iterative algorithms of various kinds are employed to execute a numerical solution. These algorithms generally solve the MPBVP by reducing it to a corresponding initial value problem and starting with a set of initial conditions $x(t_1) = (a_1, a_2, ..., a_n)'$, and employ different iterative schemes to modify the initial conditions so as to satisfy the given boundary conditions [1–6, 8, 14–17, 19–21, 24]. As the number of the ODEs and the BCs as well as the entire interval $[t_1, t_m]$ increases, however, these methods run into difficulties because of numerical errors, large computer storage requirements, and the excessive amount of computer time needed to solve simultaneously the entire system of equations along the entire interval.

In order to establish a basic existence theorem [3, 7, 9, 12, 23] guaranteeing the global solvability of the corresponding initial value problem and to mitigate the difficulties mentioned above, it is often necessary and effective

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to analyze the MPBVPs by information flows among the ODEs and the BCs before a solution is attempted.

In Section 2, we first introduce a compact method to represent the information flows among the ODEs (1.1) and the BCs (1.2), and relate these equations to digraphs and their associated Boolean matrices [10, 11, 13, 22] which represent the structure of information flows among these equations.

The theory of MPBVPs relies heavily on initial value problems. In Section 3, the initial value problem for (1.1) is discussed first from the graph theoretical point of view and compared with the usual analytical procedures. The theorems of existence for solutions of two point boundary value problems for second-order ODEs are well developed [3, 4, 12]. The existence theorem for general nonlinear MPBVPs, however, is considerably more complicated and less throughly developed than that for the second-order two-point boundary value problems. By applying the Boolean matrices, we then derive a global necessary condition for existence of the solutions for the general nonlinear MPBVPs. An algorithm to examine solvability of the MPBVP is also given.

In a large system of nonlinear MPBVPs which arises in a physical process, there exists such a possibility that a subset of the ODEs with a subset of the BCs does not contain any variables in common with the remaining equations in the system. In such a case the subset of equations, which is called a disjoint subsystem [11, 13], can be solved completely independently of the remaining equations in the system. The identification by inspection, however, of the disjoint subsystems in a large system is in general impractical. In Section 4, we first propose an algorithm which can easily be loaded on a computer to identify such subsystems by operations of the Boolean matrices.

If the ODEs with BCs in a subsystem are ordered into minimally levelled hierarchical structures such that a subset of equations with the minimum subinterval can be solved independently of the remaining equations in the sequence and influenced only by equations belonging to the higher levels in the system hierarchy, the solution can proceed seriously from higher level to lower level. Consequently, the numerical errors and the computer time for a solution as well as the computer storage requirements would be greatly reduced. With this object in mind, we propose algorithms to construct the hierarchical structure in the subsystem and to identify the minimum subinterval [19] for the multipoint boundary value subproblem in each level.

2. Representation of MPBVPs

It is very convenient to relate the ODEs and the BCs to digraphs and the associated Boolean matrices which have a one-to-one correspondence with the structure of the digraphs and can be easily performed on a computer.

2.1. *Equation and Boundary Matrices*

A digraph D_e for the ODEs is a collection of points $\dot{x}_1, \dot{x}_2, ..., \dot{x}_n$ (denoted by the set $\dot{X}$) and directed lines corresponding to the outputs of the points $x_1, x_2, ..., x_n$ (denoted by the set X) joining all or some of the points called directed paths (or simply paths). Similarly, a digraph D_b for the BCs is a collection of points $g_1, g_2, ..., g_n$ (denoted by the set G) and directed lines $x_1^l, x_2^l, ..., x_n^l$ at $t = t_l$ (denoted by the subset $X_l, X_l \subset X$; $l = 1, 2, ..., m, m \geqslant 2$) joining all or some of the directed paths.

The Boolean matrix for the ODEs is called here *equation matrix E* and defined as follows:

(i) each row of the equation matrix corresponds to a derivative with respect to t, $\dot{x}_i$, and each column corresponds to a system variable x_j,

(ii) an entry of the $n \times n$ matrix, e_{ij}, is either a Boolean 1 or 0 according to the rule

$$\begin{aligned} e_{ij} &= 1, \quad && \text{if system variable } x_j \text{ appears in the function } f_i, \\ &= 0, && \text{otherwise.} \end{aligned}$$

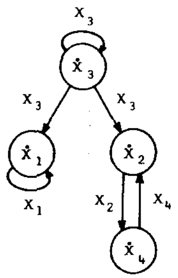
This matrix then indicates the occurrence of the system variables in each of the ODEs. For example, consider the following set of ODEs given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_3, t), \qquad & \dot{x}_2 &= f_2(x_3, x_4, t), \\ \dot{x}_3 &= f_3(x_3, t), & \dot{x}_4 &= f_4(x_2, t). \end{aligned} \tag{2.1}$$

Figure 2.1 shows the digraph and its equation matrix E corresponding to (2.1). In the figure the derivative $\dot{x}_i$ is integrated to be its output x_i. The argument x_j in a function f_i is considered to be a path from the jth point to the ith equation and its direction is indicated by an arrowhead. It is easily seen from the matrix that the equations that are descendents are indicated by the nonzero entries in the column and the equations that are predecessors by the nonzero entries in the row.

On the other hand, the Boolean matrix for the BCs is called here the *boundary matrix B* and defined as follows:

(i) each row of the $n \times n \cdot m$ matrix $B = [B_1 \mid B_2 \mid, ..., \mid B_m]$ corresponds to a boundary condition g_i,



$$E = \begin{array}{c} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

FIG. 2.1. The digraph D_e and its associated equation matrix E of ODEs (1.1).

(ii) an entry of the $n \times n$ submatrix B_l, b_{ijl}, is either Boolean 1 or 0 according to the rule

$$b_{ijl} = 1, \quad \text{if system variable at } t = t_l,\ x_j(t_l) \in X_l, \text{ appears in boundary condition } g_i \in G,$$
$$= 0, \quad \text{otherwise.}$$

For example, consider the following three-point BCs given by

$$\begin{aligned} &g_1(x_3(t_2), x_3(t_3)) = 0, \qquad g_2(x_1(t_1), x_3(t_2)) = 0, \\ &g_3(x_1(t_2), x_4(t_3)) = 0, \qquad g_4(x_4(t_2), x_4(t_3)) = 0. \end{aligned} \tag{2.2}$$

Then the boundary matrix of (2.2) is given by

$$B = [B_1 \mid B_2 \mid B_3]$$

$$= \begin{array}{c} \\ \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{array} \begin{array}{c} \begin{array}{cccc|cccc|cccc} & t_1 & & & & t_2 & & & & t_3 & & \\ x_1 & x_2 & x_3 & x_4 & x_1 & x_2 & x_3 & x_4 & x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}. \tag{2.3}$$

2.2. *Boolean Algebra*

In the subsequent discussions, we also require the Boolean algebra [10, 11, 13].

Boolean multiplication: If $a, b, c, \ldots$, are propositions, their logical product is one if all propositions are true and zero if any proposition is false. Briefly,

$$a \cdot b \cdot c \cdots = \min(a, b, c, \ldots).$$

Boolean union: If $a, b, c,...,$ are propositions, their logical sum is zero if every proposition is false, and one if any addend is one. Briefly,

$$a + b + c + \cdots = \max(a, b, c,...).$$

If there is a path $\dot{x}_i \dot{x}_j$ from $\dot{x}_j$ to $\dot{x}_i$, we say that $\dot{x}_i$ is *reachable* from $\dot{x}_j$. The number of lines in a path is called its *length*. The following statements can therefore be made concerning the digraph of Fig. 2.1: $\dot{x}_1$ is reachable from $\dot{x}_3$, but $\dot{x}_3$ is not reachable from $\dot{x}_1$; the length from $\dot{x}_3$ to $\dot{x}_4$ is two.

An important property of the equation matrix E is that the kth power of this matrix gives all the kth step paths between points, and the nonzero entry e_{ij}^k of the power matrix E^k indicates that there is a path going through k paths from $\dot{x}_j$ (or x_j) to $\dot{x}_i$. For example, the i, j entry e_{ij}^k of the second power of the matrix E is formed according to the following formula:

$$e_{ij}^2 = e_{i1} \cdot e_{1j} + e_{i2} \cdot e_{2j} + \cdots + e_{in} \cdot e_{nj}, \qquad i, j = 1, 2,..., n,$$

where n is the order of the matrix E.

The following theorem for the power matrix is useful:

THEOREM 2.1. *The i, j entry e_{ij}^k is one if and only if there exists in D_e at least one sequence of length k $(k \leqslant n-1)$ from $\dot{x}_j$ to $\dot{x}_i$.*

As to the proof, see [10, 13].

The second and third powers for Fig. 2.1 are given by

$$E^2 = E^3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \tag{2.4}$$

and the rows of E^2 and E^3 still correspond to $\dot{x}_i$ to which the flows are directed and the columns correspond to x_j (or $\dot{x}_j$) from which flows are directed.

2.3. *Reachability Matrix*

Let us now consider the reachability matrix R whose entries are denoted by r_{ij} and defined as follows:

$$\begin{aligned} r_{ij} &= 1, \qquad \text{if } \dot{x}_i \text{ is reachable from } \dot{x}_j, \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

In other words, if the digraph D_e contains a path from $\dot{x}_j$ to $\dot{x}_i$, then $r_{ij} = 1$.

In constructing the reachability matrix of the digraph, we use the fact that each point is reachable from itself. The entries on the diagonal of R are, therefore, all 1's. The following theorem interrelates the power matrix E^k and the reachability matrix R:

THEOREM 2.2. *For the digraph D_e with n points, the following relation holds*:

$$R = I + E + E^2 + \cdots + E^{n-1} = (I+E)^{n-1}, \tag{2.5}$$

where I is the $n \times n$ identity matrix.

As to the proof, see [10, 13].

The reachability matrix for (2.1) is given by

$$R = (I+E)^3 = (I+E)^2 = \begin{array}{c} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}. \tag{2.6}$$

We say that the D_e is *transitive* if for every three distinct points $\dot{x}_i, \dot{x}_j$, and $\dot{x}_k$, whenever the paths $\dot{x}_i\dot{x}_j$ and $\dot{x}_j\dot{x}_k$ are both in D_e, then the path $\dot{x}_i\dot{x}_k$ is also in D_e. There is a correspondence between the concept of reachability in a digraph and that of transitivity in a binary relation. The *transitive closure* D_e^t of the given digraph D_e is the minimal transitive digraph containing D_e and the same set of points as D_e. Theorem 2.3 interrelates the reachability matrix R and the transitive closure D_e^t.

THEOREM 2.3. *For any two distinct points $\dot{x}_i$ and $\dot{x}_j$ in D_e, the path $\dot{x}_i\dot{x}_j$ is in D_e^t if and only if $\dot{x}_j$ is reachable from $\dot{x}_i$ in D_e.*

As to the proof, see [10].

From the reachability matrix (2.6), we now have the transitive digraph of Fig. 2.1 as shown in Fig. 2.2.

2.4. *Converse and Symmetrized Matrices*

Let us now introduce the definition of the *converse* D' of a digraph D. Given a digraph D, its converse D' is the digraph with the same set of points such that for any two points u and v the line uv is in D' if and only if the line vu is in D. We now have

THEOREM 2.4. *The converse of the converse of a digraph D is D itself*; *symbolically*, $D'' = D$.

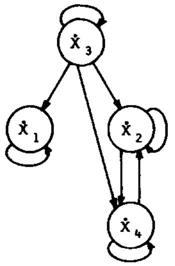


FIG. 2.2. The transitive closure D_e^t of Fig. 2.1.

As to the proof, see [10].

The converse digraph D_e' of the digraph D_e in Fig. 2.1, for example, is shown in Fig. 2.3. From the figure the converse equation matrix E' for (2.1) is given by

$$E' = \begin{array}{c} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}. \tag{2.7}$$

These observations are summarized and generalized in

THEOREM 2.5. *Let E be the equation matrix of the digraph D_e. Then in the converse equation matrix E' the i, j entry, e'_{ij}, has a nonzero element if and only if the j, i entry, e_{ji}, in the matrix E has a nonzero element.*

The theorem is obvious from the definition of the converse matrix.

In the subsequent discussions, we require one more definition. The *symmetrized digraph* of a digraph D, written D^*, is the symmetric digraph obtained from D by adding a directed line uv whenever this line does not already appear in D but the corresponding line vu does appear in D.

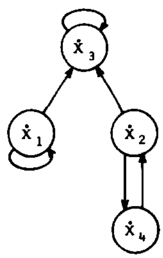


FIG. 2.3. The converse digraph D_e' of Fig. 2.1.

As shown in Fig. 2.4, we have from Figs. 2.2 and 2.3 the symmetrized digraph of Fig. 2.1. From the figure the corresponding symmetrized matrix E^* for (2.1) is given by

$$E^* = \begin{array}{c} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}. \tag{2.8}$$

From (2.8), it is easily seen that the following relation holds:

$$E^* = E + E'. \tag{2.9}$$

We now have

THEOREM 2.6. *Let E' and E^* be the converse and the symmetrized matrices respectively of the equation matrix E of* (1.1). *Then relation* (2.9) *holds.*

The theorem is obvious from the definitions of the converse and the symmetrized equation matrices.

3. Global Necessary Condition for Existence

In the previous section, the representations of nonlinear MPBVPs by Boolean matrices were introduced. In this section, by applying the results of Section 2, we first show that the initial value problem for a system of ODEs can be interrelated with its reachability matrix. From the graph theoretical point of view, we then discuss a global necessary condition for existence of the solutions for general nonlinear MPBVPs.

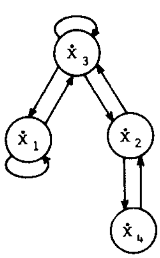


FIG. 2.4. The symmetrized digraph D_e^* of Fig. 2.1.

3.1. *Initial Value Problems*

In the system of ODEs, we considered reachability which indicates whether a point $\dot{x}_i$ can reach a point $\dot{x}_j$. Let us now consider the reachability from the initial conditions a_i to the points $\dot{x}_j$.

We have shown that the reachability matrix for example (2.1) is given by (2.6). From the matrix, we formally have the following imaginary ODEs termed here as *transitive* ODEs:

$$\begin{aligned} \dot{x}_1 &= \bar{f}_1(x_1, x_3, t), & \dot{x}_2 &= \bar{f}_2(x_2, x_3, x_4, t), \\ \dot{x}_3 &= \bar{f}_3(x_3, t), & \dot{x}_4 &= \bar{f}_4(x_2, x_3, x_4, t). \end{aligned} \tag{3.1}$$

It is easily seen from the equations that $\dot{x}_4$ in (3.1), for example, depends ultimately not only on x_2 but also on x_3 and x_4. On the other hand, $\dot{x}_2$, corresponding to the argument x_2 in $\bar{f}_4$, depends on x_2, x_3, and x_4 which are the same arguments in $\bar{f}_4$, and $\dot{x}_3$, corresponding to the argument x_3 in $\bar{f}_4$, depends on x_3 which is contained in the arguments of $\bar{f}_4$. These facts show that, once the initial conditions a_2, a_3, and a_4 for $\dot{x}_2$, $\dot{x}_3$, and $\dot{x}_4$ respectively are given, the solutions $x_2(t)$, $x_3(t)$, and $x_4(t)$ of example (2.1) can be obtained formally by integrating the corresponding equations, i.e.,

$$\begin{aligned} x_1(t) &= h_1(a_1, a_3, t), & x_2(t) &= h_2(a_2, a_3, a_4, t), \\ x_3(t) &= h_3(a_3, t), & x_4(t) &= h_4(a_2, a_3, a_4, t), \qquad t > t_1. \end{aligned} \tag{3.2}$$

Consequently, it is easily seen from (3.2) that the reachability matrix of (2.6) is equivalent to

$$\hat{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}, \tag{3.3}$$

which is termed here the *equivalent reachability matrix* for (2.1).

These observations are summarized and generalized in

THEOREM 3.1. *Assume that the initial conditions* $x_i(t_1) = a_i$ $(i = 1, 2, \ldots, n)$ *for* (1.1) *are given. If the* i, i_j *entries* $(j = 1, 2, \ldots, n_i,\ n_i \leqslant n;$ $i = 1, 2, \ldots, n)$ *of the reachability matrix* R *corresponding to* (1.1) *are nonzero, then the solutions* $x_i(t)$ *of initial value problem* (1.1) *for* $t > t_1$ *can be expressed as some functions of the initial conditions* a_{i_j}, *i.e.,*

$$x_i(t) = h_i(a_{i_1}, a_{i_2}, \ldots, a_{i_{n_i}}, t), \qquad t > t_1, \quad i = 1, 2, \ldots, n. \tag{3.4}$$

For the linear (or linearized) initial value problems, it is well known that the functions h_i are given explicitly by the i, i_j entries of the fundamental matrix for the linear equations and the associated initial conditions [7, 12].

3.2. *Multipoint Boundary Value Problems*

Let us now consider reachability from the set of ODEs to the set of BCs. For the equivalent reachability matrix $\hat{R}$ given by (3.3) of example (2.1) and the boundary submatrices B_2 and B_3 given by (2.3), we consider the following operations:

$$B_2\hat{R} = \begin{array}{c} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \cdot \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}\!\!\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} = \begin{array}{c} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}, \tag{3.5a}$$

$$B_3\hat{R} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{array}{c} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}. \tag{3.5b}$$

It is immediately apparent from (3.5) that matrices $B_2\hat{R}$ and $B_3\hat{R}$ show the reachabilities from the initial conditions a_i $(i = 1, 2, 3, 4)$ to the BCs g_i $(i = 1, 2, 3, 4)$ at $t = t_2$ and $t = t_3$, respectively. Using (2.3) and (3.5), let us now define the $n \times n$ Boolean matrix $\bar{R}$ by

$$\bar{R} = B_1 + (B_2 + B_3)\hat{R} = \begin{array}{c} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}, \tag{3.6}$$

which is termed here the *reachability matrix from the* ODEs *to the* BCs.

From the analytical point of view, the operations B_1, $B_2\hat{R}$, and $B_3\hat{R}$ show that the solutions of (3.2) at t_1, t_2, and t_3 are substituted into BCs (2.2). In fact, from (2.2) and (3.2) we have the following new BCs rewritten formally as functions of the initial conditions:

$$\begin{aligned} \bar{g}_1(a_3) &= 0, & \bar{g}_2(a_1, a_3) &= 0, \\ \bar{g}_3(a_1, a_2, a_3, a_4) &= 0, & \bar{g}_4(a_2, a_3, a_4) &= 0. \end{aligned} \tag{3.7}$$

It is easily seen that the Boolean matrix corresponding to (3.7) is equivalent to the reachability matrix $\bar{R}$ given by (3.6).

These observations are summarized and generalized in

THEOREM 3.2. *Let R (or $\hat{R}$) be the reachability matrix of the n ODE* (1.1) *and $B = [B_1 \mid B_2 \mid ,..., \mid B_m]$ be the $n \times n \cdot m$ boundary matrix for* (1.2). *Then the unique $n \times n$ reachability matrix which shows the reachabilities from the initial conditions a_i for* (1.1) *to the* BC*s g_j of* (1.2) *is given by*

$$\bar{R} = B_1 + \left(\sum_{l=2}^{m} B_l\right) \hat{R}. \tag{3.8}$$

We have thus reduced the problem of solving the MPBVP given by (2.1) and (2.2) to that of finding the roots a_i of a system of algebraic equations (3.7) [3, 12, 14, 21]. It is easily seen by inspection from (3.7) that the equation $\bar{g}_1$ contains only the initial condition a_3, and that it is formally possible to solve $\bar{g}_1$ for a_3 independently of the remaining equations. After a_3 is known, a_1 can be solved from $\bar{g}_2$. Similarly, after a_1 and a_3 are known, since the remaining two equations contains a_2 and a_4, the initial conditions a_2 and a_4 can be obtained by solving $\bar{g}_3$ and $\bar{g}_4$ simultaneously. As a result, all the initial conditions can be formally solved from (3.7). This shows that each row g_i and each column a_j in the reachability matrix $\bar{R}$ must have at least a nonzero element, respectively. In fact, the matrix $\bar{R}$ given by (3.6) has nonzero elements in each row and each column.

We say that the $n \times n$ reachability matrix $\bar{R}$ has full rank n if each row and each column has at least one nonzero element. The above discussions are summarized and generalized in the following main theorem for nonlinear MPBVPs:

THEOREM 3.3 (global necessary condition for existence). *Let $\bar{R}$ be the $n \times n$ reachability matrix from* (1.1) *to* (1.2). *If the* MPBVP *has a solution, then the matrix $\bar{R}$ defined by* (3.8) *has full rank n.*

Proof. Suppose that the ith row of $\bar{R}$ consists of all zero elements. Then it follows immediately that the dimensionality of the BCs is degenerated to $n - 1$ and one of the initial conditions cannot be determined exactly. On the other hand, suppose that the jth column of the matrix consists of all zero elements. Then it follows at once that there is no path from the corresponding initial condition a_j to any BC of (1.2). This shows that the initial condition a_j cannot be determined from (1.2).

3.3. *Algorithm for Examining the Necessary Condition*

If the global necessary condition for the nonlinear MPBVP is not satisfied, it is of no use to try a solution. Thus is would be very effective to examine the necessary condition of Theorem 2.3. The algorithm for that purpose is now summarized as follows:

Step 1. From the ODEs (1.1) and the BCs (1.2), form the equation matrix E and the boundary matrix B, respectively.

Step 2. From (2.5) construct the reachability matrix R corresponding to the equation matrix E.

Step 3. Using the submatrices B_l ($l = 1, 2, \ldots, m$) of B, construct (3.8) and obtain the reachability matrix $\bar{R}$ from the initial conditions to the BCs.

Step 4. Examine the rank of the matrix $\bar{R}$.

The algorithm can easily be loaded into a computer for the analysis and is quite efficient in examining the global necessary condition.

4. Decompositions of the Nonlinear MPBVPs

Once the necessary condition for existence of the nonlinear MPBVP is satisfied, we then proceed to the decomposition of the large MPBVP into multipoint boundary subproblems (MPBVSPs) with minimum subintervals. Here, three phases are included in the decomposition, i.e., (i) identifying the MPBVSPs, (ii) partitioning the MPBVSP into a hierarchical structure, and (iii) determining the minimum subinterval for each MPBVSP. By these analyses the numerical errors and the computer time as well as the computer storage would be greatly reduced.

4.1. *Identifying Disjoint* MPBVSP*s*

The most effective method for decomposition is to find disjoint subsystems [11, 13, 22], i.e., subsets of ODEs and the BCs that do not contain any common variables so that each subset can be treated independently. By this decomposition, the computer storage needed to effect a solution would correspond roughly to the size of the largest subsystem of MPBVSPs instead of the entire MPBVP.

For example, consider the following set of ODEs:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_4, t), & \dot{x}_2 &= f_2(x_2, t), \\ \dot{x}_3 &= f_3(x_4, x_6, t), & \dot{x}_4 &= f_4(x_4, t), \\ \dot{x}_5 &= f_5(x_2, x_5, t), & \dot{x}_6 &= f_6(x_3, t). \end{aligned} \tag{4.1}$$

The corresponding equation matrix E and the reachability matrix R are given by

$$E = \begin{array}{c|cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline \dot{x}_1 & 1 & 0 & 0 & 1 & 0 & 0 \\ \dot{x}_2 & 0 & 1 & 0 & 0 & 0 & 0 \\ \dot{x}_3 & 0 & 0 & 0 & 1 & 0 & 1 \\ \dot{x}_4 & 0 & 0 & 0 & 1 & 0 & 0 \\ \dot{x}_5 & 0 & 1 & 0 & 0 & 1 & 0 \\ \dot{x}_6 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}, \qquad R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}. \tag{4.2}$$

If the function f_i contains a variable x_j but f_j does not contain x_i, then there is a directed path only from $\dot{x}_j$ to $\dot{x}_i$. Even if the converse directed path $\dot{x}_i\dot{x}_j$ is added artificially to E, there is no change in the property that these two equations are still joined. Taking this fact into account, we form the symmetrized equation matrix E^* from (4.2) and Theorem 2.6.

$$E^* = E + E' = \begin{array}{c|cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline \dot{x}_1 & 1 & 0 & 0 & 1 & 0 & 0 \\ \dot{x}_2 & 0 & 1 & 0 & 0 & 1 & 0 \\ \dot{x}_3 & 0 & 0 & 0 & 1 & 0 & 1 \\ \dot{x}_4 & 1 & 0 & 1 & 1 & 0 & 0 \\ \dot{x}_5 & 0 & 1 & 0 & 0 & 1 & 0 \\ \dot{x}_6 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}. \tag{4.3}$$

From Theorem 2.2, we define the reachability matrix R^* for the symmetrized equation matrix E^* of (4.3) by

$$R^* = (I + E^*)^5 = \begin{array}{c|cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline \dot{x}_1 & 1 & 0 & 1 & 1 & 0 & 1 \\ \dot{x}_2 & 0 & 1 & 0 & 0 & 1 & 0 \\ \dot{x}_3 & 1 & 0 & 1 & 1 & 0 & 1 \\ \dot{x}_4 & 1 & 0 & 1 & 1 & 0 & 1 \\ \dot{x}_5 & 0 & 1 & 0 & 0 & 1 & 0 \\ \dot{x}_6 & 1 & 0 & 1 & 1 & 0 & 1 \end{array}. \tag{4.4}$$

The rows of R^* still correspond to the derivatives $\dot{x}_i$, and the columns correspond to the variables x_j. Since a subset of the derivatives, $\dot{x}_1$, $\dot{x}_3$, $\dot{x}_4$, and $\dot{x}_6$, contains the corresponding variables, x_1, x_3, x_4, and x_6, simultaneously, and another subset of derivatives, $\dot{x}_2$ and $\dot{x}_5$, contains x_2 and

x_5 simultaneously, it is easily seen from (4.4) that $\dot{x}_1$, $\dot{x}_3$, $\dot{x}_4$, and $\dot{x}_6$ constitute a disjoint subsystem of ODEs and $\dot{x}_2$ and $\dot{x}_5$ comprise the other disjoint subsystem of ODEs.

These observations are summarized and generalized in

THEOREM 4.1. *Let the matrices E and R^* be the equation matrix for* (1.1) *and the reachability matrix corresponding to the symmetrized equation matrix E^*, respectively. Then the equations $\dot{x}_i$ and $\dot{x}_j$ in* (1.1) *can be decomposed into disjoint subsystems of* ODEs *if and only if the* ith *and* jth *rows in R^* are different from each other $(i, j = 1, 2, \ldots, n;\ i \neq j)$.*

Associated with the ODEs (4.1), consider the following four-point BCs:

$$\begin{aligned} g_1(x_2(t_2), x_5(t_3)) &= 0, & g_2(x_1(t_1), x_4(t_2)) &= 0, \\ g_3(x_4(t_2), x_4(t_3)) &= 0, & g_4(x_1(t_2), x_6(t_3)) &= 0, \\ g_5(x_2(t_3), x_5(t_3), x_5(t_4)) &= 0, & g_6(x_6(t_2), x_6(t_3)) &= 0. \end{aligned} \tag{4.5}$$

The corresponding boundary matrix B is given by

$$B = [B_1 \mid B_2 \mid B_3 \mid B_4]$$

$$= \begin{array}{c|cccccc|cccccc|cccccc|cccccc|} & \multicolumn{6}{c}{t_1} & \multicolumn{6}{c}{t_2} & \multicolumn{6}{c}{t_3} & \multicolumn{6}{c}{t_4} \\ & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & & & & & & & & & & & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline g_1 & 0&0&0&0&0&0 & 0&1&0&0&0&0 & 0&0&0&0&1&0 & 0&0&0&0&0&0 \\ g_2 & 1&0&0&0&0&0 & 0&0&0&1&0&0 & 0&0&0&0&0&0 & 0&0&0&0&0&0 \\ g_3 & 0&0&0&0&0&0 & 0&0&0&1&0&0 & 0&0&0&1&0&0 & 0&0&0&0&0&0 \\ g_4 & 0&0&0&0&0&0 & 1&0&0&0&0&0 & 0&0&0&0&0&1 & 0&0&0&0&0&0 \\ g_5 & 0&0&0&0&0&0 & 0&0&0&0&0&0 & 0&1&0&0&1&0 & 0&0&0&0&1&0 \\ g_6 & 0&0&0&0&0&0 & 0&0&0&0&0&1 & 0&0&0&0&0&1 & 0&0&0&0&0&0 \\ \hline \end{array}. \tag{4.6}$$

By analogy with (3.8) in Theorem 3.2, we form the following matrix $\bar{R}^*$:

$$\bar{R}^* = B_1 + \left(\sum_{l=2}^{4} B_l\right) R^* = \begin{array}{c|c} & \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array} \\ \begin{array}{c} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}, \tag{4.7}$$

which shows the symmetrized reachabilities from the initial conditions a_i $(i = 1, 2,..., 6)$ of (4.1) to the BCs g_j $(j = 1, 2,..., 6)$ of (4.5), and is called the symmetrized reachability matrix from ODEs (4.1) to BCs (4.5). Since a subset of the BCs, g_2, g_3, g_4, and g_6 in (4.7), contains the initial conditions, a_1, a_3, a_4, and a_6, simultaneously, it is easily seen from (4.4) and (4.7) that the subset of ODEs, $\dot{x}_1$, $\dot{x}_3$, $\dot{x}_4$, and $\dot{x}_6$, with the subset of the BCs, g_2, g_3, g_4, and g_6, constitute a disjoint MPBVSP called the first MPBVSP. Another subset of the BCs, g_1 and g_5, contains a_2 and a_5 simultaneously. This shows that another subset of the ODEs, $\dot{x}_2$ and $\dot{x}_5$, with the BCs g_1 and g_5 comprise the other disjoint MPBVSP, called the second MPBVSP.

These observations are summarized and generalized in

THEOREM 4.2. *Let E^*, R^*, and $B = [B_1 | B_2 |,..., | B_m]$ be the symmetrized equation matrix for* (1.1), *the reachability matrix obtained from E^*, and the boundary matrix for* (1.2), *respectively. Similarly to* (4.7), *define the symmetrized reachability matrix $\bar{R}^*$ by*

$$\bar{R}^* = B_1 + \left(\sum_{l=2}^{m} B_l\right) R^*. \tag{4.8}$$

Suppose now that (i) *there are s sets of different row vectors in the matrix $\bar{R}^*$ and* (ii) *the i, j entry, say, in $\bar{R}^*$ is nonzero. Then the* MPBVP *given by* (1.1) *and* (1.2) *can be decomposed into s disjoint* MPBVSPs, *and the* ODE $\dot{x}_j$ *and the* BC g_i *belong to the same* MPBVSP.

4.2. *Partitioning of the* MPBVSP

Once the complete system of the MPBVP has been decomposed into the disjoint MPBVSPs, it is then desirable to partition each MPBVSP into minimally levelled hierarchical structures such that a hierarchical level comprises a number of ODEs and BCs which are independent of each other and influenced only by the ODEs and the BCs belonging to the higher levels in the subsystem hierarchy.

From the results of the previous section, the equation submatrix $E^{(1)}$ and the boundary submatrix $B^{(1)}$ for the first MPBVSP, for example, are given by

$$E^{(1)} = \begin{array}{c} \\ \dot{x}_1 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_6 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_3 & x_4 & x_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array},$$

$$B^{(1)} = [B_1^{(1)} \mid B_2^{(1)} \mid B_3^{(1)}]$$

$$= \begin{array}{c} \\ \\ g_2 \\ g_3 \\ g_4 \\ g_6 \end{array} \begin{array}{c} \begin{array}{cccc|cccc|cccc} & t_1 & & & & t_2 & & & & t_3 & & \\ x_1 & x_3 & x_4 & x_6 & x_1 & x_3 & x_4 & x_6 & x_1 & x_3 & x_4 & x_6 \end{array} \\ \left[\begin{array}{cccc|cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}. \tag{4.9}$$

From (3.8) with (4.9), we have the corresponding reachability submatrices $R^{(1)}$ (or $\hat{R}^{(1)}$) and $\bar{R}^{(1)}$ for the first MPBVSP:

$$R^{(1)} = (I + E^{(1)})^3 = \begin{array}{c} \\ \dot{x}_1 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_6 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_3 & x_4 & x_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array},$$

$$\bar{R}^{(1)} = B_1^{(1)} + (B_2^{(1)} + B_3^{(1)})\,\hat{R}^{(1)} = \begin{array}{c} \\ g_2 \\ g_3 \\ g_4 \\ g_6 \\ \text{cs} \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_3 & a_4 & a_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\ \begin{array}{cccc} 2 & 2 & 4 & 2 \end{array} \end{array} \begin{array}{c} \text{rs} \\ 2 \\ 1 \\ 4 \\ 3 \\ \\ \end{array}, \tag{4.10a}$$

where the *row sum*, written rs, shows the number of nonzero entries in each row and similarly the *column sum*, written cs, shows the number of nonzero entries in each column.

In order to partition matrix (4.10a) into a hierarchical structure, one must first establish what information each equation is to supply, that is, the identity of the initial condition whose value is to be obtained from the equations. The initial condition is called the *output variable* of the equation and the set of all of the variables assigned to the equations is called an *output set* [13, 22]. The remaining initial conditions of the equation are called *input variables*. Steward [22] proposed a method for finding an output set. The method seems, however, to be less efficient. Recently an algorithm has been proposed by the present author [18]. The algorithm for (4.10a) proceeds as follows:

(i) Since the rs of row 2 (g_3) in (4.10a) is equal to 1, it is easily seen that its corresponding initial condition a_4 is the unique output variable of g_3. Remove then the second row and the third column from $\bar{R}^{(1)}$.

(ii) We now have the following reduced matrix:

$$\begin{array}{cccc} & a_1 & a_3 & a_6 \\ g_2 & & & \\ g_4 & & & \\ g_6 & & & \end{array} \begin{array}{c} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{cccc} \text{rs} & \text{Eq.} & \text{Output} \\ 1 & g_3 & a_4 \\ 3 & & \\ 2 & & \end{array} \tag{4.10b}$$

$$\begin{array}{cccc} \text{cs} & 2 & 2 & 2 \end{array}$$

Since the rs of the first row in (4.10b) is 1, the corresponding initial condition a_1 is the output variable of g_2. Remove the first row and the first column.

(iii) We lastly have the following reduced matrix:

$$\begin{array}{c|cc|cccc} & a_3 & a_6 & \text{rs} & \text{Eq.} & \text{Output} \\ g_4 & 1 & 1 & 2 & g_3 & a_4 \\ g_6 & 1 & 1 & 2 & g_2 & a_1 \\ \text{cs} & 2 & 2 & & & \end{array} \tag{4.10c}$$

Since all entries of the matrix consists of nonzero elements, we can arbitrarily assign a_3 and a_6 to the output variable g_4 and g_6, respectively.

From the above procedures, we hve the following output set for the first MPBVSP:

$$g_2—a_1, \qquad g_3—a_4, \qquad g_4—a_3, \qquad g_6—a_6. \tag{4.10d}$$

Replacing the variables a_1, a_3, a_4, and a_6 in (4.10a) by the corresponding equations g_2, g_4 g_3, and g_6, respectively, and permuting the columns so that they have the same sequence as the rows, we have the following Boolean matrix $\bar{\bar{R}}^{(1)}$ for the first MPBVSP:

$$\bar{\bar{R}}^{(1)} = \begin{array}{c} \\ g_2 \\ g_3 \\ g_4 \\ g_6 \end{array} \begin{array}{c} \begin{array}{cccc} g_2 & g_3 & g_4 & g_6 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \end{array}. \tag{4.11}$$

Let us now determine the smallest groups of BCs in a subsystem that must be solved simultaneously. From the matrix $\bar{\bar{R}}^{(1)}$, compute the reachability matrix $\tilde{R}^{(1)}$ which shows the reachabilities from the BC g_i to the BC g_j.

$$\tilde{R}^{(1)} = \bar{\bar{R}}^{(1)3} = \begin{array}{c} \\ g_2 \\ g_3 \\ g_4 \\ g_6 \\ \text{cs} \end{array} \begin{array}{c} \begin{array}{cccc} g_2 & g_3 & g_4 & g_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\ \begin{array}{cccc} 3 & 4 & 2 & 2 \end{array} \end{array} \begin{array}{ccc} \text{rs} & \text{ds} & \text{Level} \\ 2 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & -2 & 3 \\ 4 & -2 & 3 \\ & & \end{array} \tag{4.12}$$

Since the third and fourth rows in the matrix $\tilde{R}^{(1)}$ have the same row vectors, it is easily seen that g_4 and g_6 constitute a group that must be solved simultaneously.

Let us define ds by

$$\text{ds}(g_i) = \text{cs}(g_i) - \text{rs}(g_i). \tag{4.13}$$

Then the value of ds shows the difference between the number of points which may be reached from g_i and the number of points from which g_i is reachable [18]. For example, g_3 in the second row of the matrix $\tilde{R}^{(1)}$ does not obtain any information from the remaining equations but g_3 is feeding information a_4 to g_2, g_4, and g_6. It follows that g_3 with the maximal number ds $= 3$ belongs to the highest level, i.e., the first level. Similarly g_2 with ds $= 2$ belongs to the second level, and g_4 and g_6 with the minimal number ds $= -2$ belong to the lowest level, i.e., the third level.

From the results of Sections 4.1 and 4.2, we have Fig. 4.1 in which the information flows between the ODEs given by (4.1) and the BCs given by (4.5) are shown. Table I also shows the hierarchical structures obtained for the MPBVP. As for the INT in the table, see Subsection 4.3.

4.3. *Determining the Minimum Subintervals*

We have shown thus far how an entire MPBVP can be decomposed into a number of disjoint MPBVSPs and then how each MPBVSP can be rearranged into a hierarchical structure.

Let us now consider the algorithm to determine the minimum subintervals for which the MPBVPs in each level must be solved [19]. It can easily be seen from boundary matrix (4.6) that, for example, there are two paths from x_2 at t_2 to g_1 and from x_2 at t_3 to g_5. Therefore the ODE $\dot{x}_2$ must be solved at least for the subinterval $[t_2, t_3]$. In order to determine the minimum subintervals from the equation and the boundary matrices, let us first form the

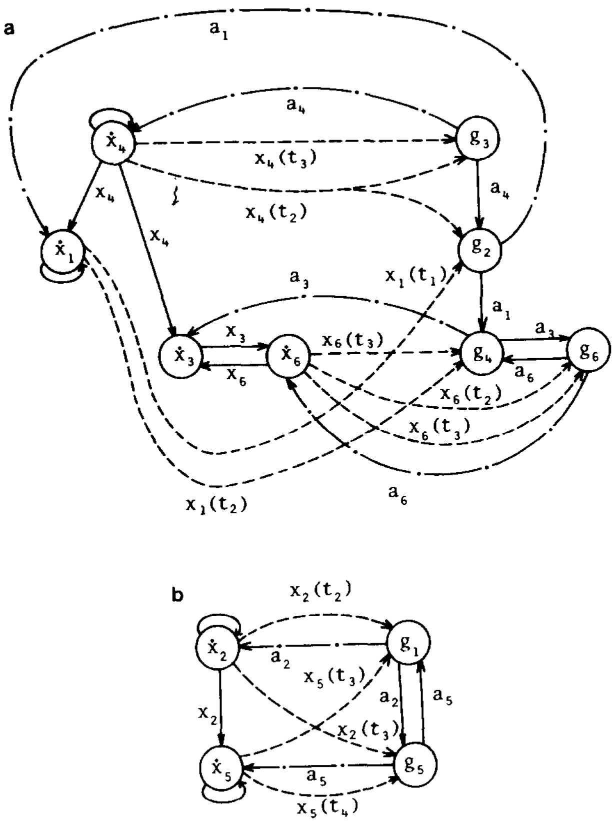


FIG. 4.1. The information flows of the MPBVP given by (4.1) and (4.5), (a) the first MPBVSP; (b) the second MPBVSP.

TABLE I

The Hierarchical Structures for MPBVP (4.1) and (4.5)

First MPBVSP				Second MPBVSP			
Level	ODE	BC	INT	Level	ODE	BC	INT
1	$\dot{x}_4$	g_3	$[t_1, t_3]$	1	$\dot{x}_2$	g_1	$[t_2, t_4]$
2	$\dot{x}_1$	g_2	$[t_1, t_2]$		$\dot{x}_5$	g_5	$[t_3, t_4]$
3	$\dot{x}_3$ $\dot{x}_6$	g_4 g_6	$[t_2, t_3]$				

Boolean vector c by adding all rows (the boundary conditions) in the boundary matrix B

$$c = \sum_{i=1}^{n} g_i \begin{array}{c} t_1 \quad t_2 \quad \ldots \quad t_m \\ x' \quad x' \quad \ldots \quad x' \\ [c_1 \mid c_2 \mid, \ldots, \mid c_m], \end{array} \tag{4.14}$$

which is called here the *condensed boundary vector*. The jth entry c_{jl} at $t = t_l$ of the $1 \times n$ condensed boundary subvector c_l is either 1 or 0 according to the following Boolean union:

$$\begin{aligned} c_{jl} &= 1, \qquad \text{if} \quad b_{1jl} + b_{2jl} + \cdots + b_{njl} = 1, \quad j = 1, 2, \ldots, n, \quad l = 1, 2, \ldots, m, \\ &= 0, \qquad \text{otherwise,} \end{aligned}$$

where $c_l = (c_{1l}, c_{2l}, \ldots, c_{nl})$ and b_{ijl} is the i, j entry of the boundary submatrix B_l.

The condensed boundary vector c for (4.6) is given by

$$c = \sum_{i=1}^{6} g_i \begin{array}{cccccccccccccc} & & & t_1 & & & & & & & t_2 & & & \\ & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & & & & & & & \\ {[}1 & & 0 & 0 & 0 & 0 & 0 & \mid & 1 & 1 & 0 & 1 & 0 & 1 \quad \mid \end{array}$$

$$\begin{array}{cccccccccccccc} & & & t_3 & & & & & & & t_4 & & & \\ & & & & & & & & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \mid & 0 & 1 & 0 & 1 & 1 & 1 & \mid & 0 & 0 & 0 & 0 & 1 & 0]. \end{array} \tag{4.15}$$

Then, form the $n \times m$ matrix C, the *condensed boundary matrix*, by transposing the subvector c_i of (4.14):

$$C = x \begin{array}{c} t_1 \quad t_2 \quad \cdots \quad t_m \\ {[}c_1' \quad c_2' \quad \cdots \quad c_m'], \end{array} \tag{4.16}$$

which shows the relation between x and t obtained simply from the boundary matrix B independently of the ODEs.

The condensed boundary matrix C for the example is

$$C = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}. \tag{4.17}$$

It is easily seen from (4.17) that the apparent subinterval for x_2 is $[t_2, t_3]$ and the subinterval for x_3 is undetermined.

Let us now interelate the condensed boundary matrix C with the reachability matrix R by the following Boolean multiplication:

$$T = \overset{\tau}{\dot{x}}[R'C], \qquad \tau = (t_1, t_2, \ldots, t_m), \tag{4.18}$$

where the $n \times m$ matrix T is called the *interval matrix.* It is obvious from the definitions of the reachability matrix R and the condensed boundary matrix C that the interval matrix T shows the reachability from t_j to $\dot{x}_i$. As to the interval matrix, we have

THEOREM 4.3. *If the* MPBVP *given by* (1.1) *and* (1.2) *satisfies the necessary condition of Theorem* 3.3, *then every row of the interval matrix* T *defined by* (4.18) *contains at least one nonzero entry.*

The proof of the theorem is obtained readily from Theorem 3.3.

From (4.18), the minimum subinterval for $\dot{x}_i$, denoted by INT $\dot{x}_i$, can now be obtained according to the following rule:

(i) if row i of the interval matrix T has nonzero entries greater than or equal to two, and the least and the greatest numbers of the columns with nonzero entries are j and k, respectively, then the minimum interval for $\dot{x}_i$ is given by INT $\dot{x}_i = [t_j, t_k]$;

(ii) if row i of T has only one nonzero entry in column j $(1 \leqslant j \leqslant m-1)$, then INT $\dot{x}_i = [t_j, t_{j+1}]$,

(iii) if row i of T has only one nonzero entry in the last column m, then INT $\dot{x}_i = [t_{m-1}, t_m]$.

The interval matrix for the example is obtained from (4.2) and (4.17) as

$$T = \begin{array}{c} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}. \tag{4.19}$$

Applying (i)–(iii), we have from (4.19) the following minimum subintervals:

$$\begin{aligned} &\mathrm{INT}\,\dot{x}_1 = [t_1, t_2], \qquad &&\mathrm{INT}\,\dot{x}_4 = [t_1, t_3], \\ &\mathrm{INT}\,\dot{x}_2 = [t_2, t_4], &&\mathrm{INT}\,\dot{x}_5 = [t_3, t_4], \\ &\mathrm{INT}\,\dot{x}_3 = [t_2, t_3], &&\mathrm{INT}\,\dot{x}_6 = [t_2, t_3], \end{aligned} \tag{4.20}$$

and these results are shown in Table I.

From the table, it is easily seen that (i) the entire MPBVP given by (4.1) and (4.5) is disjointed into two MPBVSPs; (ii) the first MPBVSP is partitioned further into three levels: (A) at the first level, the ODE $\dot{x}_4$ with the BC g_3 is solved for the subinterval $[t_1, t_3]$, (B) at the second level, since the solution x_4 is known, $\dot{x}_1$ with g_2 is solved for the subinterval $[t_1, t_2]$, and (C) at the third level, since the solutions x_1 and x_4 are known, $\dot{x}_3$ and $\dot{x}_6$ with g_4 and g_6 are solved simultaneously for the subinterval $[t_2, t_3]$; (iii) the second MPBVSP can be decomposed into two stages: (A) for the subinterval $[t_2, t_3]$, $\dot{x}_2$ is solved, (B) for the subinterval $[t_3, t_4]$, $\dot{x}_2$ and $\dot{x}_5$ with g_1 and g_5 are solved simultaneously; and (iv) thus the total number of subintervals that must be solved is decreased from 18 to 8.

Note that once the multipoint boundary value problems in the subintervals determined from the above procedure are solved by a numerical method, one can easily obtain the solutions in the remaining subintervals by computing the original differential equations backwards or forwards from the known boundary conditions.

5. Concluding Remarks

In this paper several practical methods by graph theory have been proposed to analyze the structures of large nonlinear multipoint boundary value problems. By these analyses, a global necessary condition for existence of the solution was derived first, and then three decomposition algorithms were presented. These methods can easily be programmed on a computer and the general subroutine, labelled MPTUAI, is developed for these analyses.

Using an existing language for algebraic and symbolic manipulations, e.g., REDUCE 2 [25], the equation matrix E and the boundary matrix B can easily be generated automatically from the given ordinary differential equations and the boundary conditions, respectively. Further, it is easily seen from (4.8) that the computer storage requirements for the boundary matrix could be greatly reduced if one gives the matrix in the form $B = [B_1 \mid \sum_{l=2}^{m} B_l]$.

Moreover, using MPTUAI with an existing numerical method for the solution of nonlinear multipoint boundary value problems, e.g., the initial value adjusting method (labelled MPJUND) in [19], it would be possible to reduce greatly the numerical errors, the computer storage requirements, and the excessive amount of computer time for the solution.

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